Lecture 23 - Landau theory

Behavior of systems during phase transitions (esp. at the critical point) displays universality (ex: scaling laws near critical points)

This occurs for different materials
Ex: gas → liquid transition ^3He, CO_2, etc.

And for completely different physical phenomena!
Ex: gas → liquid vs. paramagnet → ferromagnet
vs. normal → superconductor

All display same behavior (provided there are commonalities (like dimensionality))

Landau theory (1936) of phase transitions
Formulation of general theory for 2nd-order phase transitions

- Phenomenological - not derived from microscopic theory or first principles (in most cases)
- Mean-field theory

Key pts: system minimizes Gibbs free energy G (or Helmholtz free energy F if $V = \text{const.}$) during phase transition

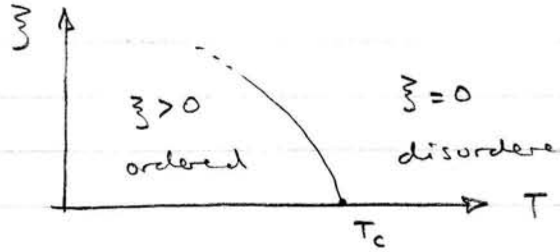
a minimum with respect to what variable(s)?

Order parameter ξ

Phase transitions involve transition
from disordered (high entropy S) phase $\rightarrow$ ordered (low S) phase

Ex: gas $\longrightarrow$ liquid
paramagnet $\rightarrow$ ferromagnet
$(\uparrow\downarrow\uparrow\uparrow\downarrow)$ $(\uparrow\uparrow\uparrow\uparrow\uparrow)$
normal $\longrightarrow$ superconductor etc...

ξ is a single parameter that specifies phase



In disordered phase $\xi = 0$
ordered phase $\xi > 0$

Ex: Ferromagnet $\xi = M$ magnetization
Liquid-gas $\xi = \rho - \rho_c$ liquid density $\left(\rho_c = \text{critical density}\right)$
Superconductor $\xi = |\Psi|^2$ fraction of superconducting electrons
$\vdots$

Landau postulated a Landau free energy

$F_L(\xi, T)$ At equilibrium $\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0$

If there is more than 1 minimum
lowest determines equilibrium state $\xi_{eq}(T)$

Now we are interested in behavior near critical pt.
so ξ is small. Expand $F_L(\xi, T)$ in powers of ξ

Near T_c:

$$F_L(\xi,T) = g_0(T) + g_1(T)\,\xi + \frac{g_2(T)}{2}\xi^2 + \frac{g_3(T)}{3}\xi^3 + \frac{g_4(T)}{4}\xi^4 \ldots$$

where $g_i(T)$ are functions of T — highest order we'll need

Look at equilibrium ξ:

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = g_1(T) + g_2(T)\,\xi + g_3(T)\,\xi^2 + g_4(T)\,\xi^3 = 0$$

But for $T > T_c$, solution must be $\xi = 0$ *
so $g_1 = 0$

Can use symmetry arguments to remove odd orders $g_3, g_5 \ldots$
Ex: in ferromagnetism $\xi = M$. Below T_c, two opposite spin states $M = \uparrow\uparrow\uparrow \ldots$ $-M = \downarrow\downarrow\downarrow \ldots$ are equally likely, so we expect F_L to be <u>even</u> in ξ

$$F_L(\xi,T) = g_0(T) + \frac{1}{2} g_2(T)\,\xi^2 + \frac{1}{4} g_4(T)\,\xi^4$$

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 = g_2(T)\,\xi + g_4(T)\,\xi^3$$

$$\Rightarrow \xi_0 = 0 \quad \text{or} \quad \xi_\pm = \pm\sqrt{\frac{-g_2(T)}{g_4(T)}}$$

* This is true for ferromagnet <u>if</u> $B = 0$
we will see how to generalize F_L for $B \neq 0$

Plugging in those sol'ns back into $F_L(\xi, T)$

$$F_L(\xi, T) = g_0(T) \qquad \text{for } \xi_0 = 0 \text{ sol'n}$$
$$= g_0(T) - \frac{1}{4}\frac{g_2(T)^2}{g_4(T)} \qquad \text{for } \xi_\pm = \pm\sqrt{\frac{-g_2}{g_4}} \text{ sol'ns}$$

Now we expect $\xi_0 = 0$ to be sol'n for $T \geq T_c$
but expect $\xi_\pm = \pm\sqrt{\frac{-g_2}{g_4}}$ to be sol'ns for $T < T_c$

Notice that since $\xi_\pm = \pm\sqrt{\frac{-g_2(T)}{g_4(T)}}$, when $g_2(T) > 0$ these sol'ns are <u>imaginary</u>

So $\xi_0 = 0$ is only physical solution

On the other hand, when $g_2(T) < 0$, these sol'ns will be <u>real</u> and F_L is minimized (since $F_L < g_0(T)$)

So $g_2(T) > 0$ for $T \geq T_c$
$g_2(T) < 0$ for $T < T_c$

In general, we are interested in behavior near T_c, so we expand all $g_i(T)$ to lowest orders in $T - T_c$

$$g_i(T) = g_i^{(0)} + g_i^{(1)}(T - T_c) + \cdots$$

For $g_2(T)$, above condition means that $g_2^{(0)} = 0$

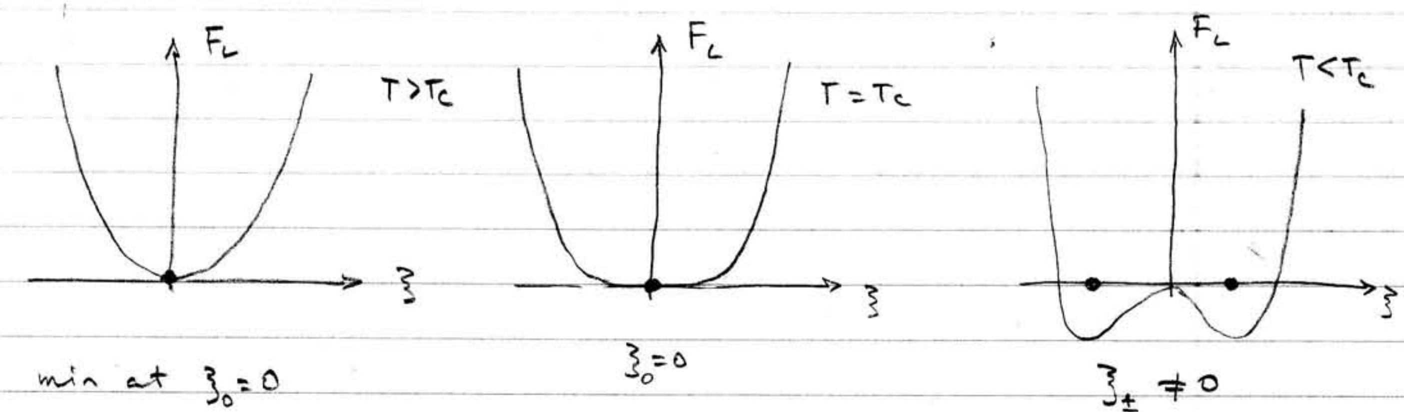
$$g_2(T) = g_2^{(1)}(T - T_c) \quad \text{or} \quad \alpha(T - T_c)$$

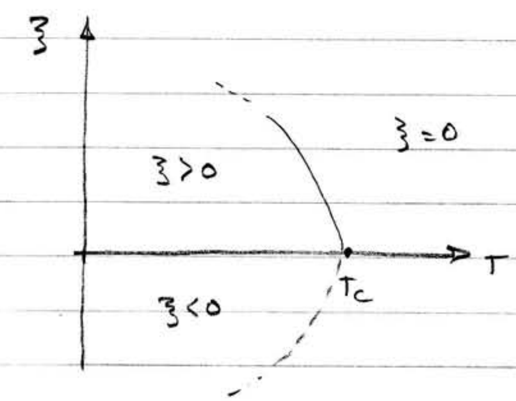
(For g_0, g_4 we cannot make that statement)

So, to lowest order

$$F_L = g_0(T) + \frac{1}{2}\alpha(T-T_c)\,\xi^2 + \frac{1}{4}g_4\,\xi^4$$

Plot this vs. ξ (ignore offset $g_0(T)$ which doesn't change to leading order)





For $T<T_c$ sol'ns are

$$\xi = \pm\sqrt{\frac{-g_2(T)}{g_4(T)}} = \pm\sqrt{\frac{T_c-T}{g_4}}$$

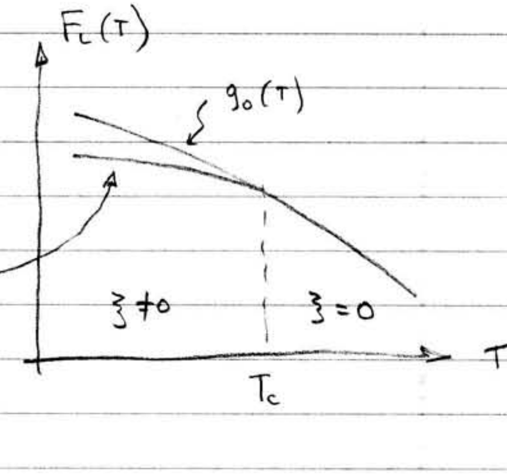
$$\therefore\ \xi \sim \left(1-\frac{T}{T_c}\right)^{\beta} \qquad \beta=\frac{1}{2}$$

We get the same critical exponent

At equilibrium value of ξ

$$F_L(T) = g_0(T) \qquad T \geq T_c$$
$$= g_0(T) - \frac{\alpha^2}{4}\frac{(T-T_c)^2}{g_4} \qquad T<T_c$$

System adopts minimum F_L



This matches the behavior of the ferromagnet (where $\xi = M$) and also a fluid near the critical pt. (where $\xi = \rho - \rho_c$)

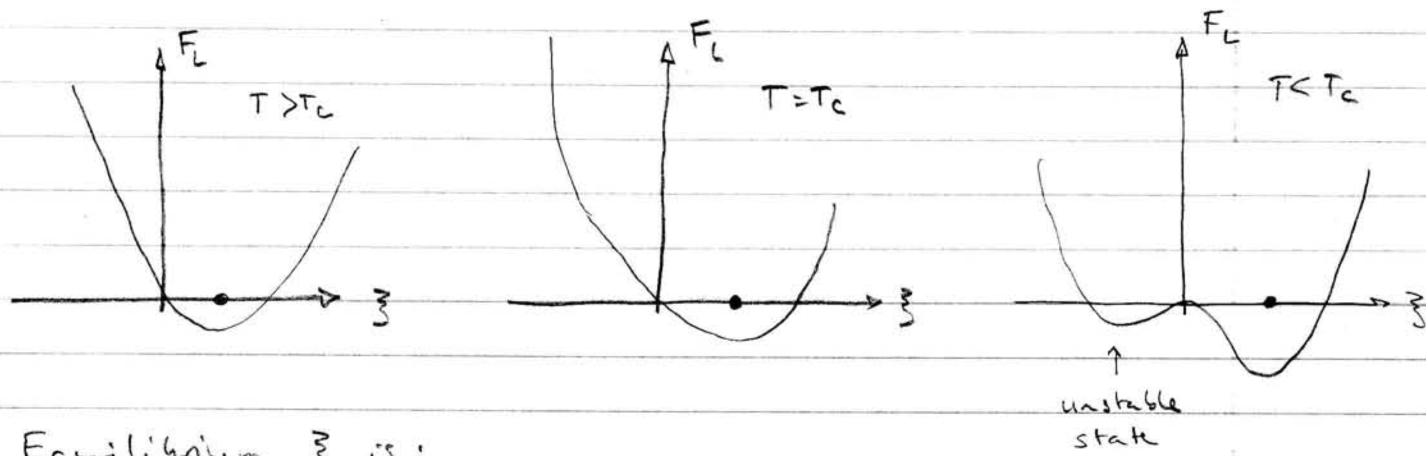
Landau's postulate is that all second order phase transitions will have $F_L(\xi, T)$ of this form $\Rightarrow$ universal behavior!

In ferromagnet, we had a magnetic field. How do we incorporate this in Landau theory. Simple: in magnet case add energy $-MB$

$$\therefore \boxed{F_L(\xi, T) = g_0(T) + \frac{\alpha}{2}(T - T_c)\xi^2 + \frac{g_4}{4}\xi^4 - H\xi}$$

H = magnetic field for ferromagnet, for fluid H = pressure ...
(like ξ, H represents different quantity in different system)

Replot F_L vs. ξ for $H > 0$



Equilibrium ξ is:

$$\frac{\partial F_L}{\partial \xi} = 0 = \alpha(T - T_c)\xi + g_4\xi^3 - H$$

To lowest order (ignore ξ^3 term) $\quad \xi = \dfrac{H}{\alpha(T - T_c)}$

$$\therefore \chi = \left(\frac{\partial \xi}{\partial H}\right)_T \sim (T - T_c)^{-\gamma} \qquad \gamma = 1$$

as before

Summary of Landau theory

- General theory of second order phase transitions

$$F_L = g_0 + \frac{1}{2}\alpha(T-T_c)\xi^2 + \frac{1}{4}g_4\xi^4 - H\xi \quad \text{valid near } T_c$$

$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0$ determines equilibrium value for ξ

- leads to universal scaling laws near T_c with critical exponents $\Rightarrow$ we saw β, γ but there are others $(\alpha, \beta, \gamma, \delta, \nu, \eta)$

- Landau theory gets lots of things qualitatively right
However, because it is a mean field theory, it does not get the critical exponent values exactly right
Ex: predicts $\beta = 1/2$ $\gamma = 1$, experimentally $\beta = 0.33$, $\gamma = 1.25$

- to get better agreement requires a lot of work and advanced theoretical techniques

- Landau theory is phenomenological, but in some cases can be connected directly to microscopic theory
Ex: Weiss ferromagnet, van der Waals models can be expanded near T_c and put in the exact form $F_L(\xi)$ above

Extension to first order transitions
(Note: the formalism I used is different from K+K)

Previously, we excluded odd terms in F_L : $g_3\xi^3$ ...
What if we put it back in?

$$F_L(\xi, T) = g_0(T) + \frac{1}{2}\alpha(T-T_c)\xi^2 + \frac{1}{3}g_3\xi^3 + \frac{1}{4}g_4\xi^4 \quad (\text{Let } H=0$$

$$\left(\frac{\partial F_L}{\partial \xi}\right)_T = 0 = \alpha(T-T_c)\xi + g_3\xi^2 + g_4\xi^3.$$

$$0 = \xi\left[\alpha(T-T_c) + g_3\xi + g_4\xi^2\right]$$

$\therefore$ Sol'ns are $\xi_0 = 0$ or $\xi_\pm = \dfrac{-g_3 \pm \sqrt{g_3^2 - 4g_4\alpha(T-T_c)}}{2g_4}$

$\xi_\pm$ are <u>real</u> sol'ns when $g_3^2 - 4g_4\alpha(T-T_c) \geq 0$

i.e at a temp $T_c^* = T_c + \dfrac{g_3^2}{4g_4\alpha}$

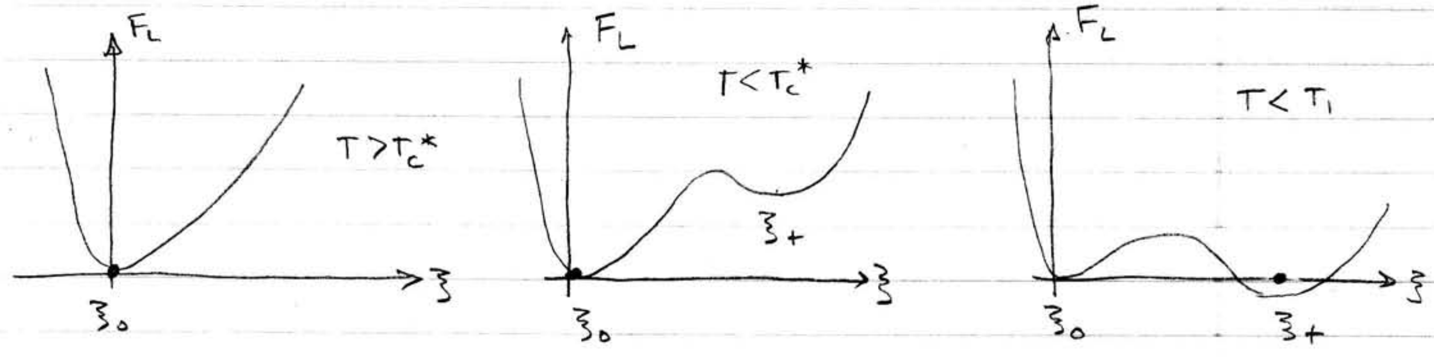
So, for $T > T_c^*$ $\quad \xi_0 = 0$ is only sol'n

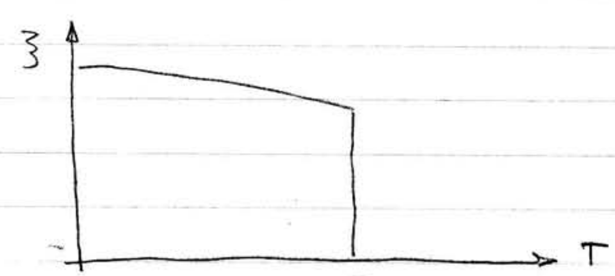
$T \lesssim T_c^*$ $\quad$ there are 2 local minima, but absolute minimum still at $\xi_0 = 0$

At some $T \leq T_1$ $\quad$ Absolute minimum is at ξ_+

Equilibrium ξ jumps <u>discontinuously</u> from $\xi_0 \to \xi_+$ at T_1

$\Rightarrow$ First order phase transition.





This is what happens in gas $\to$ liquid transition away from critical pt.

Critical exponent	**Definition***	**Fluid (expt.)**	**Magnet (expt.)**	**3-d Ising model (theo.)**	**Mean field**
α	$C \sim \lvert T - T_C \rvert^{-\alpha}$	0.11 – 0.116	0.16	0.110	0
β	$\xi \sim \lvert T - T_C \rvert^{\beta}$	0.316 – 0.327	0.33 ± 0.03	0.325 ± 0.0015	1/2
γ	$\chi \sim \lvert T - T_C \rvert^{-\gamma}$	1.23 – 1.25	1.33 ± 0.03	1.2405 ± 0.0015	1
δ	$H \sim \lvert \xi \rvert^{\delta}$ at T_C	4.6 – 4.9	4.1 ± 0.1	4.82(4)	3

* For a magnet $\xi = M$, $\chi = \left(\frac{\partial M}{\partial H}\right)_T$

For a fluid $\xi = \rho - \rho_C$, $\chi \rightarrow K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$, $H \rightarrow P - P_C$

Values from Nigel Goldenfeld “Lectures on Phase Transitions and the Renormalization Group” and Eugene Commins “Statistical Mechanics Lectures” (UC Berkeley).